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Bayes Estimation under Balanced Loss Functions

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Abstract

In this research, the researcher has derived different standard Bayes estimators for scale parameter of Exponential distribution by using balanced and unbalanced loss function. This estimation includes two cases availability and lack of primary information (Jeffery and Gamma conjugate priors) about the phenomenon studied. Simulation experiments with different sample sizes and virtual parameters have been built for this purpose, then a comparison is mads between these estimators depend on the Mean Square Error (MSE) criteria. The results have demonstrating the superiority of balanced loss functions in the absence of prior information about the phenomenon studied, while they may not have the same efficiency if availability of prior information about this phenomenon is cannot found.
Keywords: Exponential Distribution, Bayes Method, Balanced Loss Functions .

المستخلص

تم في هذا البحث إيجاد مقدرات طريقة بيز القياسية لمعلمة القياس للتوزيع الأسي بإستخدام دوال الخسارة المتزنة وغير المتزنة ، وذلك في حالتي توفر وعدم معلومات مسبقة (دالة كاما المرافقة الطبيعية ومعلومات فيشر على التوالي) عن الظاهرة المدروسة . تم بناء تجارب محاكاة بأحجام عينات ومعلمات افتراضية مختلفة لهذا الغرض ، ثم أجريت مقارنة بين هذه المقدرات بالإعتماد على معيار متوسط مربعات الخطأ (MSE). أثبتت النتائج أفضلية تقديرات دوال الخسارة المتزنة في حالة غياب (أو قلة) المعلومات المسبقة حول الظاهرة المدروسة ، في حين أنها قد لا تكون بنفس الفعالية إذا ما توفرت المعلومات المسبقة حول تلك الظاهرة.
الكلمات المفتاحية : التوزيع الأسي ، طريقة بيز القياسية ، دوال الخسارة المتزنة .

1- Introduction & aim:

The exponential distribution is one of the important and commonly used distributions in studies that deal with the failure's times, survival, and waiting queues. Many researchers have dealt with estimation process for this distribution by using Maximum Likelihood method, which

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is based on the observable data. (Until the advent of Bayes method, which relyies heavily on the prior distributions and loss functions) .In this area, (Al-Kutubi, 2009) was employment an extension Jeffery prior distribution to estimate the exponential distribution parameter, furthermore, proposed a new loss function by using simulation. (Oayd, 2012:1) compared the estimators of the Bayes method for the parameter and reliability and failure rate of Rayleigh distribution using balanced and unbalanced loss functions. The main problem of estimating process with Byes method lies in the absence of primary information about the phenomenon studied, which always leads us to resort to the non-informative prior. This prior may not always give us the best estimation. Therefore, this research aims to avoid this problem through dealing with the balanced loss functions, and to achieve to our aim we focus on Bayes method using simulation study to estimate the exponential distribution parameter under balanced and unbalanced loss functions, different known priors have been use, and then comparing these estimators based on the (MSE).

2- Standard Bayes Method:

Let $(z_1, z_2, \dots, z_n)$ be a random sample of size (n) from Exponential distribution, and then the probability density function $f(z;\alpha)$ is:

$$f(z;\alpha) = \begin{cases} \alpha\, e^{-\alpha z} & \alpha, z > 0 \\ 0 & o.w \end{cases} \qquad (1)$$

And the cumulative function for (z) can is:

$$F(z;\alpha) = Pr(Z \leq z) = \int_0^z f(z)dz = 1 - e^{-\alpha z} \qquad (2)$$

The standard Bayes method assumes that the parameters to be estimate are random variables, this parameter should be present in a probability density function known as prior distribution, then by combining the Likelihood function and this prior by using the Bayes inversion formula we can obtain a probability density function called the Posterior distribution, according to the following formula:

$$f(\alpha|z) = \frac{L(z|\alpha)\, f(\alpha)}{\int_{\forall\theta} L(z|\alpha)\, f(\alpha) d\alpha} \qquad (3)$$

Where:

$L(z|\alpha)$: likelihood function for sample $z_1, z_2, \dots, z_n$.

$f(\alpha)$: Prior distribution .

$f(\alpha|z)$: posterior distribution.

The likelihood function for can evaluate as:

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$$L(z_1, z_2, \dots, z_n|\alpha) = \prod_{i=1}^{n} f(z_i|\alpha) = \prod_{i=1}^{n} \alpha \ e^{-\alpha \sum_{i=1}^{n} z_i} = \alpha^n \ e^{-\alpha \sum_{i=1}^{n} z_i} \qquad (4)$$

2-1 Prior Distributions:

Priors can be divided into two types according to the abundance of primary information as:

1- Non-Informative Prior:

If there is no sufficient initial information about the estimated parameter, or not yet available, the prior distribution can be selected depending on the Jeffery's formula that is based on the parameter's zone(α). Since the parameter field is positive$(0, \infty)$, then we can find the prior distribution by using either Regular Logarithmic distribution follow:

a- Regular Logarithmic prior:

This prior distribution can be define as:

$$f_r(\alpha) \propto \frac{1}{\alpha} = \frac{c}{\alpha} \qquad c\ , \alpha > 0 \qquad (5)$$

Where c is a constant.

b- Fisher information prior:

Fisher information $I\ (\alpha)$ is definitely as:

$$f_q(\alpha) \propto \sqrt{I(\alpha)} = c\ .\sqrt{I(\alpha)} \qquad , c \ is \ a \ constsnt \qquad (6)$$

$$I(\alpha) = -nE\left[\frac{\partial^2 \ln f(z;\alpha)}{\partial \alpha^2}\right] = \frac{n}{\alpha^2} \qquad (7)$$

Substitute (7) in (56 we get the Fisher information prior distribution:

$$f_q(\alpha) = c\ .\ \frac{\sqrt{n}}{\alpha} \qquad (8)$$

Note that we will assume that c=1.

2- Informative Prior (Conjugate Prior):

Conjugate prior is a known specific and appropriate probability function of the parameter, thus the conjugate prior for exponential distribution follows the Gamma distribution with parameters (θ, β), and it has the following p.d.f:

$$f_2(\alpha) = \begin{cases} \dfrac{\beta^\theta}{\gamma(\theta)} \alpha^{\theta-1} e^{-\beta\alpha} & , \theta, \beta, \alpha > 0 \\ 0 & o.w \end{cases} \qquad (9)$$

2-2 Posterior Distribution:

The posterior distribution for the parameter (α) that is identified in (3) differs according to the prior distribution that used. It can be evaluate by using the three mentioned priors as follows:

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1- Regular Logarithmic distribution posterior $f_r(\alpha|z)$, compensating (4) and (5) in (3), as follows:

$$f_r(\alpha|z)=\frac{\alpha^{-1}\,\alpha^{n}\;e^{-\alpha\sum_{i=1}^{n}z_i}}{\int_0^\infty \alpha^{-1}\,\alpha^{n}\;e^{-\frac{\sum_{i=1}^{n}z_i}{\alpha}}d\alpha}=\frac{\alpha^{(n-1)}\;e^{-\alpha\sum_{i=1}^{n}z_i}}{\int_0^\infty \alpha^{(n-1)}\,e^{-\alpha\sum_{i=1}^{n}z_i}\,d\alpha}$$

Since:

$$\int_0^\infty \alpha^{(n-1)}\,e^{-\alpha\sum_{i=1}^{n}z_i}\,d\alpha=\frac{\Gamma(n)}{(\sum_{i=1}^{n}z_i)^n}$$

Where$\Gamma(n)$: represent Gamma function,$\Gamma(n)=(n-1)!$ then:

$$f_r(\alpha|z)=\frac{\alpha^{n-1}\;e^{-\alpha\sum_{i=1}^{n}z_i}}{\frac{\Gamma(n)}{(\sum_{i=1}^{n}z_i)^n}}$$

$$f_r(\alpha|z)=\frac{(\sum_{i=1}^{n}z_i)^n}{\Gamma(n)}\;\alpha^{n-1}\,e^{-\alpha\sum_{i=1}^{n}z_i} \qquad (10)$$

2- Fisher information posterior $f_q(\alpha|z)$, by compensating (4) and (8) in (3), as follows:

$$f_q(\alpha|z)=\frac{c\frac{\sqrt{n}}{\alpha}\;\alpha^{n}\;e^{-\alpha\sum_{i=1}^{n}z_i}}{\int_0^\infty c\frac{\sqrt{n}}{\alpha}\;\alpha^{n}\;e^{-\alpha\sum_{i=1}^{n}z_i}\,d\alpha}=\frac{c\sqrt{n}\;\alpha^{(n-1)}\;e^{-\alpha\sum_{i=1}^{n}z_i}}{c\sqrt{n}\int_0^\infty \alpha^{(n-1)}\,e^{-\alpha\sum_{i=1}^{n}z_i}\,d\alpha}$$

$$f_q(\alpha|z)=\frac{\alpha^{(n-1)}\;e^{-\alpha\sum_{i=1}^{n}z_i}}{\int_0^\infty \alpha^{(n-1)}\,e^{-\alpha\sum_{i=1}^{n}z_i}\,d\alpha}$$

but $\int_0^\infty \alpha^{(n-1)}\,e^{-\alpha\sum_{i=1}^{n}z_i}\,d\alpha=\frac{\Gamma(n)}{(\sum_{i=1}^{n}z_i)^n}$, then:

$$f_q(\alpha|z)=\frac{\alpha^{n-1}\;e^{-\alpha\sum_{i=1}^{n}z_i}}{\frac{\Gamma(n)}{(\sum_{i=1}^{n}z_i)^n}}$$

$$f_q(\alpha|z)=\frac{(\sum_{i=1}^{n}z_i)^n}{\Gamma(n)}\;\alpha^{n-1}\,e^{-\alpha\sum_{i=1}^{n}z_i} \qquad (11)$$

Since we assume that c=1, that leads us to $f_r(\alpha|z)=f_q(\alpha|z)$, so we will refer to the two non-informative priors as $f_1(\alpha|z)$

3- Gamma conjugate prior can be used to find the posterior distribution $f_2(\alpha|z)$, by substitute (4), (9) in (3) as follows:

$$f_2(\alpha|z)=\frac{\frac{\beta^\theta}{\Gamma(\theta)}\;\alpha^{\theta-1}\,e^{-\beta\alpha}\;\alpha^{n}\;e^{-\alpha\sum_{i=1}^{n}z_i}}{\int_0^\infty \frac{\beta^\theta}{\Gamma(\theta)}\;\alpha^{\theta-1}\,e^{-\beta\alpha}\;\alpha^{n}\;e^{-\alpha\sum_{i=1}^{n}z_i}d\alpha}$$

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$$f_2(\alpha|z) = \frac{\frac{\beta^{\theta}}{\Gamma(\theta)}\ \alpha^{n+\theta-1}\ \ e^{-\alpha(\beta+\sum_{i=1}^{n} z_i)}}{\frac{\beta^{\theta}}{\Gamma(\theta)}\int_0^{\infty}\ \alpha^{n+\theta-1}\ e^{-\alpha(\beta+\sum_{i=1}^{n} z_i)}\,d\alpha} = \frac{\alpha^{n+\theta-1}\ \ e^{-\alpha(\beta+\sum_{i=1}^{n} z_i)}}{\int_0^{\infty}\ \alpha^{n+\theta-1}\ e^{-\alpha(\beta+\sum_{i=1}^{n} z_i)}\,d\alpha}$$

$$since\ \int_0^{\infty}\ \alpha^{n+\theta-1}\ e^{-\alpha(\beta+\sum_{i=1}^{n} z_i)}\,d\alpha = \frac{\Gamma(n+\theta)}{(\beta+\sum_{i=1}^{n} z_i)^{n+\theta}}\ , then:$$

$$f_2(\alpha|z) = \frac{(\beta+\sum_{i=1}^{n} z_i)^{n+\theta}}{\Gamma(n+\theta)}\ \ \alpha^{n+\theta-1}\ e^{-\alpha(\beta+\sum_{i=1}^{n} z_i)} \qquad (12)$$

2-3 Loss Functions:

The loss function is one of the measures of accuracy in the Bayesian estimation process, loss function is defined as the amount of loss resulting under Bayes decision around unknown parameter. It is a measure of the difference between the estimated value and the real value of this parameter, i.e., $(\hat{\alpha}-\alpha)$ it should have a real non-negative value and it is usually symbolized$L(\hat{\alpha},\alpha)$.

A mathematical expectation of the loss function is called the Risk function and the amount that makes this risk function is known as minimum as possible is the standard Bayes estimator $(\hat{\alpha}_B)$ for the parameter(α) , $(\hat{\alpha}_B)$ can be calculate as follows:

$$\hat{\alpha}_B = Risk(\hat{\alpha},\alpha) = E\{L(\hat{\alpha},\alpha)\} = \int_{\forall\alpha} L(\hat{\alpha},\alpha)f(\alpha|z)d\alpha \qquad (13)$$

2-3-1 Unbalanced loss function:

Typically, loss functions are classified according to symmetry criteria into two main types: the first one is a Symmetric loss functions that supposed that the amount of loss achieved in the positive direction is equal to the amount of loss achieved in the negative direction, in the other word$\{\mathbf{L}(\hat{\boldsymbol{\alpha}},\boldsymbol{\alpha}) = |\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}|\}$. The Squared Error loss function is one of the most common symmetric loss functions, (sometimes called a Quadratic loss function). It is define as:

$$\boldsymbol{L(\hat{\alpha},\alpha) = v(\hat{\alpha}-\alpha)^2}$$

Almost researchers are assuming that$\boldsymbol{v} = \mathbf{1}$, then the quadratic loss function will become

$$\boldsymbol{L(\hat{\alpha},\alpha) = (\hat{\alpha}-\alpha)^2} \qquad (\mathbf{14})$$

Now , it is possible to obtain a standard Byes method estimator for $(\boldsymbol{\alpha})$ and under the error loss function $(\hat{\boldsymbol{\alpha}}_{\mathbf{BS}})$ through (13):

$$\boldsymbol{\hat{\alpha}_{BS} = R(\hat{\alpha},\alpha) = E(\hat{\alpha}-\alpha)^2 = \int_{\forall\alpha} (\hat{\alpha}-\alpha)^2 f(\alpha|z)d\alpha}$$

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$$\hat{\alpha}_{BS} = E(\alpha|z) = \int_{\forall\alpha} \alpha\, f(\alpha|z)\, d\alpha \qquad (15)$$

The second type of loss functions is called Asymmetric loss functions, in this type we assume that the positive and the negative directions amount of loss under Bayes decision are not necessary to be equal between, one of these functions is the Entropy loss function, which is known as the following:

$$L(\hat{\alpha},\alpha) = \left(\frac{\hat{\alpha}}{\alpha}\right) - Ln\left(\frac{\hat{\alpha}}{\alpha}\right) - 1 \qquad (16)$$

Where the standard Bayes estimator under the entropy loss function ($\hat{\alpha}_{BE}$) for the parameter (α) obtained as:

$$\hat{\alpha}_{BE} = R(\hat{\alpha},\alpha) = E\left\{\left(\frac{\hat{\alpha}}{\alpha}\right) - Ln\left(\frac{\hat{\alpha}}{\alpha}\right) - 1\right\} = \int_{\forall\alpha} \left\{\left(\frac{\hat{\alpha}}{\alpha}\right) - Ln\left(\frac{\hat{\alpha}}{\alpha}\right) - 1\right\} f(\alpha|z)\, d\alpha$$

$$\hat{\alpha}_{BE} = [E(\alpha^{-1} \mid z)]^{-1} = \left\{\int_{\forall\alpha} \alpha^{-1} f(\alpha|z)\, d\alpha\right\}^{-1} \qquad (17)$$

2-3-2 Bayes estimators under unbalanced loss functions:

Now we will find Bayes estimators under unbalanced loss functions sequentially as follows:

1- Using equations (10), (15), to find a Bayes estimator with quadratic loss function and Logarithmic prior ($\hat{\alpha}_{BS1}$):

$$\hat{\alpha}_{BS1} = E(\alpha|z) = \int_0^\infty \alpha\, f_1(\alpha|z)\, d\alpha = \int_0^\infty \frac{(\sum_{i=1}^n z_i)^n}{\Gamma(n)}\, \alpha\, \alpha^{n-1} e^{-\alpha\sum_{i=1}^n z_i}\, d\alpha$$

$$\hat{\alpha}_{BS1} = \frac{(\sum_{i=1}^n z_i)^n}{\Gamma(n)} \int_0^\infty \alpha^n e^{-\alpha\sum_{i=1}^n z_i}\, d\alpha$$

From Gamma, function properties:

$$\int_0^\infty \alpha^n e^{-\alpha\sum_{i=1}^n z_i}\, d\alpha = \frac{\Gamma(n+1)}{(\sum_{i=1}^n z_i)^{n+1}}$$

$$\hat{\alpha}_{BS1} = \frac{(\sum_{i=1}^n z_i)^n}{\Gamma(n)} * \frac{\Gamma(n+1)}{(\sum_{i=1}^n z_i)^{n+1}} = \frac{n}{\sum_{i=1}^n z_i} \qquad (18)$$

Clearly, that ($\hat{\alpha}_{BS1}$) is equal to the MLE.

2- From equations (10), (17) we can obtain Bayes estimator under Logarithmic prior and unbalanced Entropy loss function ($\hat{\alpha}_{BE1}$):

$$\hat{\alpha}_{BE1} = \left[E(\alpha^{-1} \mid z)\right]^{-1} = \left\{\int_0^\infty \alpha^{-1} f(\alpha|z)\, d\alpha\right\}^{-1}$$

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$$\hat{\alpha}_{BE1} = \left\{ \int_0^\infty \frac{(\sum_{i=1}^n z_i)^n}{\Gamma(n)} \alpha^{-1} \alpha^{n-1} e^{-\alpha \sum_{i=1}^n z_i} d\alpha \right\}^{-1}$$

$$\hat{\alpha}_{BE1} = \left\{ \frac{(\sum_{i=1}^n z_i)^n}{\Gamma(n)} \int_0^\infty \alpha^{n-2} e^{-\alpha \sum_{i=1}^n z_i} d\alpha \right\}^{-1}$$

$$Since \int_0^\infty \alpha^{n-2} e^{-\alpha \sum_{i=1}^n z_i} d\alpha = \frac{\Gamma(n-1)}{(\sum_{i=1}^n z_i)^{n-1}} \quad , then:$$

$$\hat{\alpha}_{BE1} = \left\{ \frac{(\sum_{i=1}^n z_i)^n}{\Gamma(n)} * \frac{\Gamma(n-1)}{(\sum_{i=1}^n z_i)^{n-1}} \right\}^{-1} = \left\{ \frac{\sum_{i=1}^n z_i}{n-1} \right\}^{-1} = \frac{n-1}{\sum_{i=1}^n z_i} \qquad (19)$$

3- The Bayes estimator under conjugate prior and unbalanced quadratic loss functions($\hat{\alpha}_{BS3}$) can be pediment by using equations (12),(15):

$$\hat{\alpha}_{BS2} = E(\alpha|z) = \int_0^\infty \alpha f_2(\alpha|z) d\alpha$$

$$\hat{\alpha}_{BS2} = \int_0^\infty \frac{(\beta + \sum_{i=1}^n z_i)^{n+\theta}}{\Gamma(n+\theta)} \alpha \, \alpha^{n+\theta-1} e^{-\alpha(\beta + \sum_{i=1}^n z_i)} d\alpha$$

$$\hat{\alpha}_{BS2} = \frac{(\beta + \sum_{i=1}^n z_i)^{n+\theta}}{\Gamma(n+\theta)} \int_0^\infty \alpha^{n+\theta} e^{-\alpha(\beta + \sum_{i=1}^n z_i)} d\alpha$$

$$\hat{\alpha}_{BS2} = \frac{(\beta + \sum_{i=1}^n z_i)^{n+\theta}}{\Gamma(n+\theta)} \cdot \frac{\Gamma(n+\theta+1)}{(\beta + \sum_{i=1}^n z_i)^{n+\theta+1}}$$

$$\hat{\alpha}_{BS2} = \frac{n+\theta}{\beta + \sum_{i=1}^n z_i} \qquad (20)$$

4- From equations (12), (16), it is easy to drive Bayes estimator under conjugate prior and unbalanced Entropy loss function ($\hat{\alpha}_{BE2}$) as follows:

$$\hat{\alpha}_{BE2} = [E\{f(\alpha^{-1} \mid z)\}]^{-1} = \left\{ \int_{\forall \alpha} \alpha^{-1} f(\alpha|z) d\alpha \right\}^{-1}$$

$$\hat{\alpha}_{BE2} = \left\{ \int_0^\infty \frac{(\beta + \sum_{i=1}^n z_i)^{n+\theta}}{\Gamma(n+\theta)} \alpha^{-1} \alpha^{n+\theta-1} e^{-\alpha(\beta + \sum_{i=1}^n z_i)} d\alpha \right\}^{-1}$$

$$\hat{\alpha}_{BE2} = \left\{ \frac{(\beta + \sum_{i=1}^n z_i)^{n+\theta}}{\Gamma(n+\theta)} \int_0^\infty \alpha^{n+\theta-2} e^{-\alpha(\beta + \sum_{i=1}^n z_i)} d\alpha \right\}^{-1}$$

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$$\hat{\alpha}_{BE2} = \left\{ \frac{(\beta + \sum_{i=1}^{n} z_i)^{n+\theta}}{\Gamma(n+\theta)} \cdot \frac{\Gamma(n+\theta-1)}{(\beta + \sum_{i=1}^{n} z_i)^{n+\theta-1}} \right\}^{-1}$$

$$\hat{\alpha}_{BE2} = \left\{ \frac{(\beta + \sum_{i=1}^{n} z_i)}{(n+\theta-1)} \right\}^{-1}$$

$$= \frac{n+\theta-1}{\beta + \sum_{i=1}^{n} z_i} \qquad (21)$$

2-3-3 Balanced loss function:

The symmetry criterion that is described above is not the unique criterion by which loss functions are categorized, but there is another more comprehensive criterion put down by (Zellner, 1994), which is the Balanced criterion or equilibrium criterion. The objective of achieving equilibrium in the loss function is to increase accuracy and conformity in the estimation process. The loss functions that are discussed above are consider an unbalanced loss function.

Apart from the symmetry criterion, the loss function can be a balanced according to Zellner's formula as follows:

$$L_{L,w,\alpha_0}(\hat{\alpha}, \alpha) = w\, L(\hat{\alpha}, \alpha_0) + (1-w)\, L(\hat{\alpha}, \alpha) \qquad (22)$$

Where:

$L_w(\hat{\alpha}, \alpha)$: Balanced loss function.

w: weighted coefficient,$w \in (0, 1)$.

α_0 :Primary estimator for the parameter (α)depends on the observations.

$L(\hat{\alpha}, \alpha)$: Unbalanced loss function.

$L(\hat{\alpha}, \alpha_0)$: Unbalanced loss function for the Likelihood function.

Clearly that the balanced loss function heavily depends on the weighted coefficient (w), and the initial estimator(α_0).

Lemma:

For estimating (α) under balanced loss function L_{L,w,α_0} and for a prior(α) , the Bayes estimator depends on $L(\hat{\alpha}, \alpha)$ and $f^*(\alpha|z)$, where:

$$f^*(\alpha|z) = w_{\alpha_0}(\alpha) + (1-w)\ E\{f(\alpha \mid z)\} \qquad (23)$$

According to the above lemma, the Bayes estimators using balanced loss function are different from the Bayes estimators using unbalanced loss function, because $f^*(\alpha|z)$ is difference from $f(\alpha| z)$, actually, $f^*(\alpha|z)$is a mixture of (α_0) and $f(\alpha \mid z)$.

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2-3-3 Bayes estimators under balanced loss functions:

The general formula of the balanced quadratic loss function based on the equation (15) will be computed as:

$$L^{S}_{L,w,\alpha_0}(\hat{\alpha},\alpha) = w\,L(\hat{\alpha} - \alpha_0) + (1-w)\,L(\hat{\alpha} - \alpha)$$

When (w = 0), i.e., there are no differences between Bayes estimators under the balanced and unbalanced loss functions. Now for finding the Bayes estimators we can depend on the previous lemma as:

$$\hat{\alpha}_{BBS} = E_{f^*}(\alpha|z)$$

$$\hat{\alpha}_{BBS} = w_{\alpha_0}(\alpha) + (1-w)\,E\{f(\alpha \mid z)\} \tag{24}$$

Also for balanced entropy loss function estimator is based on the equation (16) and previous lemma will be:

$$\hat{\alpha}_{BBE} = \left[E_{f^*}(\alpha^{-1} \mid z)\right]^{-1}$$

$$\hat{\alpha}_{BBE} = \left[w_{\alpha_0}(\alpha)^{-1} + (1-w)\,E\{f(\alpha^{-1} \mid z)\}\right]^{-1} \tag{25}$$

From the equations (4), (18), (19), (24) and (25), one can get the following different Bayes estimators under the balanced loss functions and non-informative prior:

1- Bayes estimator under balanced quadratic loss function($\hat{\alpha}_{BBS2}$):

$$\hat{\alpha}_{BBS1} = w\left\{\frac{n}{\sum_{i=1}^{n} z_i}\right\} + (1-w)\left\{\frac{n-1}{\sum_{i=1}^{n} z_i}\right\} \tag{26}$$

2- Bayes estimator under balanced entropy loss function ($\hat{\alpha}_{BBE2}$) :

$$\hat{\alpha}_{BBE1} = \left[w\left\{\frac{n}{\sum_{i=1}^{n} z_i}\right\}^{-1} + (1-w)\left\{\frac{n-2}{\sum_{i=1}^{n} z_i}\right\}^{-1}\right]^{-1} \tag{27}$$

Moreover, from equations (4), (20), (21), (24) and (25), we get the following different Bayes estimators under the balanced loss functions and Gamma conjugate prior:

1- Bayes estimator under Conjugate prior and balanced quadratic loss functions($\hat{\alpha}_{BBS2}$) :

$$\hat{\alpha}_{BBS2} = w\left\{\frac{n}{\sum_{i=1}^{n} z_i}\right\} + (1-w)\left\{\frac{n+\theta}{\beta + \sum_{i=1}^{n} z_i}\right\} \tag{28}$$

2- Bayes estimator under balanced entropy loss functions($\hat{\alpha}_{BBE2}$) :

$$\hat{\alpha}_{BBE2} = \left[w\left\{\frac{n}{\sum_{i=1}^{n} z_i}\right\}^{-1} + (1-w)\left\{\frac{n+\theta-1}{\beta + \sum_{i=1}^{n} z_i}\right\}^{-1}\right]^{-1} \tag{29}$$

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3- Simulation and results:

In this simulation study, we have chosen samples sizes $(n = 10,25,50,100)$, several parameter values $(\alpha = 10,2,1,0.2)$, also $(\theta = 2, \beta = \alpha)$, also we select $(w = 0.5)$ in order to override the aligned in the estimation process ,i.e., this weighed value will give the same loss to the initial estimator and Bayes estimator in their customize formulas. The number of replications used was (K=1000). The simulation program written by using (R3.5.1) program. After the parameter estimated, the Mean Square Error (MSE) calculated to compare between estimators, where:

$$MSE(\hat{\alpha}) = \frac{\sum_{i=1}^{K}\left(\hat{\theta}_i - \theta\right)^2}{K} \qquad (30)$$

The results of the simulation study are summarized and tabulated in table (1) and (2). These tables include parameter estimators, (MSE) of these estimators respectively, that is for all sample sizes and (θ) values. It is obvious from these tables:

1- The estimated values of the parameters are very close to the real values as the sample size increase.
2- When (α) is increasing، the estimated parameter values will pull away from the real values.
3- In the case of the non-informative prior, the estimated values of the parameter under the balanced loss functions are closer to the real values than these estimators that are estimated by the unbalanced loss functions, but the opposite is true in the case of the conjugate prior.
4- The (MSE) decreases as sample size increases.
5- The (MSE) increase, when (α) increase.
6- The balanced Bayes estimators with the non-informative prior gives smaller (MSE) values than the unbalanced one.
7- In the case of the conjugate prior, the unbalanced Bayes estimators have smaller (MSE) values than that the balanced one have.

4- Conclusions:

We conclude through the experimental part results that the balanced loss functions provide an efficient Bayesian estimator in the absence (or if it is having little information) of information about the studied phenomenon, i.e., in the case of Jeffery prior or non-informative prior. While in the case of the conjugate prior, balanced loss functions may not be as efficient.

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Table (1)
Parameters estimated of exponential distribution

α, β	n	$\hat{\alpha}_{BS1}$	$\hat{\alpha}_{BBS1}$	$\hat{\alpha}_{BE1}$	$\hat{\alpha}_{BBE1}$	$\hat{\alpha}_{BS2}$	$\hat{\alpha}_{BBS2}$	$\hat{\alpha}_{BE2}$	$\hat{\alpha}_{BBE2}$
0.1	10	0.1217216	0.11563552	0.136936799	0.12324312	0.099624533	0.104586986	0.108681309	0.109115374
	25	0.124128161	0.121645598	0.129525038	0.124344036	0.114039847	0.116601441	0.118425995	0.118794515
	50	0.10735369	0.106280153	0.109590225	0.10739842	0.103083284	0.10414495	0.105104525	0.10515557
	100	0.107274759	0.106738385	0.1083694	0.107285706	0.105100011	0.105651012	0.106140606	0.106171309
0.5	10	0.634151472	0.602443898	0.713420406	0.642078365	0.517280271	0.544008298	0.56430575	0.567521037
	25	0.601444212	0.589415328	0.62759396	0.602490202	0.553135596	0.56526102	0.574410042	0.575898243
	50	0.590509727	0.58460463	0.602812013	0.590755773	0.566057243	0.572378	0.577156	0.577928
	100	0.51623197	0.51365081	0.521499643	0.516284647	0.505951	0.50851	0.51096	0.511015
1	10	1.508920699	1.433474664	1.697535786	1.527782207	1.215023857	1.286526243	1.325480572	1.3417546
	25	1.149113	1.12613036	1.19907403	1.151111069	1.05847047	1.080809289	1.099180873	1.10116449
	50	1.059284969	1.048692119	1.081353406	1.059726338	1.017403144	1.027751207	1.037352	1.037726
	100	1.075467413	1.070090076	1.08644157	1.075577154	1.05363994	1.059176339	1.064072019	1.064392379
5	10	5.682602878	5.398472734	6.392928238	5.753635414	5.095285492	5.104814041	5.103947809	5.1091452
	25	5.314219249	5.207935	5.54527226	5.323461	5.02004674	5.06084861	5.097740846	5.099695662
	50	5.556690528	5.501123623	5.672454914	5.559005815	5.332266074	5.388911396	5.436820311	5.441188514
	100	5.380825631	5.353921503	5.435732015	5.381374694	5.271585661	5.299301518	5.323779579	5.325398476

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Table (2)
MSE of estimated parameter of exponential distribution

α, β	n	$\hat{\alpha}_{BS1}$	$\hat{\alpha}_{BBS1}$	$\hat{\alpha}_{BE1}$	$\hat{\alpha}_{BBE1}$	$\hat{\alpha}_{BS2}$	$\hat{\alpha}_{BBS2}$	$\hat{\alpha}_{BE2}$	$\hat{\alpha}_{BBE2}$
0.1	10	0.001671936	0.001444577	0.002564435	0.001740351	0.001200249	0.00122114	0.001275473	0.001283198
	25	0.001150161	0.001036525	0.001439721	0.001160625	0.00076511	0.000843601	0.00090751	0.000921227
	50	0.000275445	0.000260809	0.000313341	0.000276105	0.000230875	0.000238549	0.000247425	0.000247949
	100	0.000165711	0.000158195	0.000182836	0.00016587	0.000138799	0.000144723	0.000150496	0.000150874
0.5	10	0.050570613	0.043068748	0.078122265	0.052760257	0.032872603	0.034510726	0.036709225	0.037133086
	25	0.023625932	0.021330105	0.029615223	0.023839246	0.016158396	0.017594005	0.018871859	0.019095547
	50	0.014889874	0.013855806	0.017268173	0.014934473	0.011061422	0.011936494	0.012650974	0.012770631
	100	0.002875399	0.002798267	0.003074157	0.002877112	0.002647	0.002684	0.002732	0.002733
1	10	0.443424453	0.37232446	0.670980348	0.462978234	0.230659435	0.266521464	0.290361778	0.301220382
	25	0.070912	0.064586298	0.088307899	0.071511985	0.052096226	0.055207571	0.058514275	0.058911684
	50	0.025067709	0.023923924	0.028171378	0.025120237	0.021855871	0.022323131	0.022948	0.022976
	100	0.017031463	0.016248751	0.018808277	0.017048038	0.014213375	0.014837971	0.015441356	0.015482511
5	10	3.081596702	2.774430533	4.555899089	3.183616351	2.624729338	2.626635996	2.62645516	2.627562688
	25	1.139807241	1.08431	1.338395342	1.145701	1.041475376	1.044776058	1.050626777	1.05101273
	50	0.902986103	0.844206644	1.04527737	0.905569261	0.703482503	0.744333833	0.783893743	0.787729064
	100	0.428799302	0.409031571	0.47363353	0.429217799	0.357529912	0.37335254	0.388604357	0.38965531

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