استخدام نظرية المجموعات المضببة في تطوير و تكوين لوحات السيطرة (دراسة مقارنة)

A fuzzy set theory in the development and construction of control charts (Comparative study)

أ.م.د. عبد الرحيم خلف راهي*

المستخلص:

تعد السيطرة الاحصائية على العملية الانتاجية احد المفاهيم التي يتم من خلالها استخدام الاساليب الاحصائية للرقابة على الانتاج. شيوارت في العشرينات طور لوحات السيطرة الاحصائية التي تعتبر من الاساليب المهمة في ضبط الجودة لتشخيص و اكتشاف الانحرافات غير العشوائية إن وجدت. ومن اللوحات التي تستخدم بشكل واسع، لوحة المتوسط $\overline{X}$ و لوحة المدى **R** . حيث تسمى هذه اللوحات لوحات السيطرة للمتغيرات، التي تتكون عادةً من حد وسطي و حدين اعلى و ادنى للسيطرة متمثلاً بقيم عددية.

العملية الانتاجية اما تكون تحت السيطرة أو خارجها بالاعتماد على قيم المشاهدات العددية التي يمكن احتسابها، و في كثير من الحالات احتساب حدود السيطرة هذه قد لايكون دقيقاً و مؤكداً، وهنا يمكن استخدام نظرية المجموعات المضببة للتعامل مع حالات عدم التأكد و الضبابية. حيث يمكن من خلالها تحويل حدود السيطرة العددية الى حدود سيطرة مضببة بأستخدام دالة عنصر الانتماء وهذه الحدود تعطي تقييم دقيق و مرونة اكثر في الحكم و اتخاذ القرار على العملية الانتاجية مقارنةً بلوحات السيطرة الكلاسيكية المشار اليها اعلاه. في هذا البحث تم استخدام بيانات من شركة آلا للمشروبات الغازية في مدينة السليمانية في تصميم لوحة السيطرة الضبابية.

Abstract:

Statistical process control (SPC) is an approach that uses statistical techniques to monitor the process. Shewhart in 1920's developed statistical control charts that are one of the most important techniques of quality control to detect if assignable cause exists. The widely used control charts are X-bar and R charts. These are called traditional variable control chart with center line, upper control limit and lower control limit are represented by numeric values. A process is either "in control" or "out of control" depending on numeric observation values. For many problems control limits could not be so precise. Uncertainty comes from the measurement system including operators and gauges, and environmental conditions. In this context fuzzy set theory is useful tool to handle this uncertainty. Numeric control limits can be transformed to fuzzy control limits by using membership function. Fuzzy control limits provide a more accurate and flexible evaluation. In this paper through a real illustrative data from Ala Company for soft drinks in the city of sulaimani, shows the designing of fuzzy control chart for process average of variable quality.

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* أستاذ مساعد/الجامعة المستنصرية/كلية الإدارة والإقتصاد/قسم الإحصاء

مقبول للنشر بتاريخ **2011/2/23**

1- Introduction:

Control charts, proposed by Shewhart in 1920's. These charts were designed to control a process for shifts in mean and variance of a single quality characteristic. As normality is a usual assumption of control chart as well as the independency of mean and variance of normal distribution. Two main types of control charts are variable and attribute control charts. The fuzzy set theory is a more suitable tool for handling attribute data since these data may be expressed in linguistic terms such as "very good", "good", "medium", "bad", and "very bad". The fuzzy set theory was first introduced by Zadeh(1965). Many studies were done to combine statistical methods and fuzzy set theory. The fuzzy numbers are a reasonable way to analyze and evaluate the process. Some measures of central tendency in descriptive statistics are used in variable control charts. These measures can be used to convert fuzzy sets into scalars which are fuzzy mode, a-level fuzzy midrange, fuzzy median and fuzzy average.

2-1- Fuzzy $\tilde{\bar{\bar{X}}}$ and $\tilde{\bar{R}}$ control charts:

The x-bar chart is most widely used chart for controlling the process mean quality level as well as the process variability can be controlled by either a control chart for the range, called R-chart or a control chart for the standard deviation, called S-chart. In this connection fuzzy $\tilde{\bar{\bar{X}}}$ - $\tilde{\bar{R}}$ control chart are introduced.

Construct $\tilde{\bar{\bar{X}}}$ control chart based on sample ranges is given as follows:

$$UCL_{\bar{X}} = \bar{\bar{X}} + A_2 \bar{R} \quad (1)$$

$$CL_{\bar{X}} = \bar{\bar{X}} \quad (2)$$

$$LCL_{\bar{X}} = \bar{\bar{X}} - A_2 \bar{R} \quad (3)$$

where A2 is a control chart coefficient, and R is the average of Ri's that are the ranges of samples. In the fuzzy case, each sample is represented by a triangular fuzzy number (a, b, c) as shown in Fig.1.

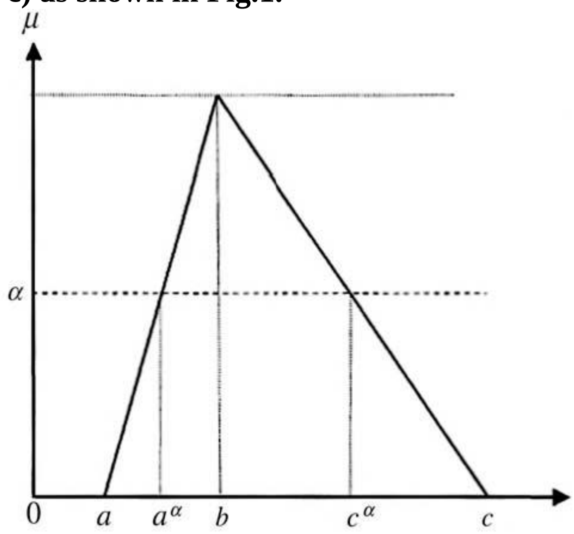

Fig.1. Representation of a sample by triangular fuzzy numbers

In this study, triangular fuzzy numbers are represented as (X_a, X_b, X_c) for each fuzzy observation. The center line $\tilde{CL}$ is the arithmetic mean of fuzzy samples, where:

$$U\tilde{CL}_{\bar{X}} = (U\tilde{C}L_1, U\tilde{C}L_2, U\tilde{C}L_3) = \tilde{CL} + A_2\bar{R} = (\bar{\bar{X}}_a, \bar{\bar{X}}_b, \bar{\bar{X}}_c) + A_2(\bar{R}_a, \bar{R}_b, \bar{R}_c)$$
$$= (\bar{\bar{X}}_a + A_2\bar{R}_a,\ \bar{\bar{X}}_b + A_2\bar{R}_b,\ \bar{\bar{X}}_c + A_2\bar{R}_c) \qquad (4)$$

$$\tilde{CL}_{\bar{X}} = (\bar{\bar{X}}_a, \bar{\bar{X}}_b, \bar{\bar{X}}_c) = \left(\frac{\sum_{j=1}^{m}\bar{X}aj}{m}, \frac{\sum_{j=1}^{m}\bar{X}bj}{m}, \frac{\sum_{j=1}^{m}\bar{X}cj}{m}\right) \qquad (5)$$

$$L\tilde{CL}_{\bar{X}} = (L\tilde{C}L_1, L\tilde{C}L_2, L\tilde{C}L_3) = \tilde{CL} - A_2\bar{R} = (\bar{\bar{X}}_a, \bar{\bar{X}}_b, \bar{\bar{X}}_c) - A_2(\bar{R}_a, \bar{R}_b, \bar{R}_c)$$
$$= (\bar{\bar{X}}_a - A_2\bar{R}_c,\ \bar{\bar{X}}_b - A_2\bar{R}_b,\ \bar{\bar{X}}_c - A_2\bar{R}_a) \qquad (6)$$

where $\bar{R}a$, $\bar{R}b$, and $\bar{R}c$ are the arithmetic means of the least possible values, the most possible values, and the largest possible values, respectively. Firstly, Raj; Rbj; Rcj are calculated as follows:

R_{aj} = X max.aj –X min.cj , R_{bj} =X max.bj – Xmin.bj , and
R_{cj} = Xmax.cj –Xmin.aj , j =1,2,... . . ,m.

where (Xmax.aj , Xmax.bj , Xmax,cj) is the maximum fuzzy number in the sample and (Xmin,aj , Xmin.bj , Xmin.cj) is the minimum fuzzy number in the sample. Then,

$$\bar{R}a = \frac{\sum Raj}{m},\quad \bar{R}b = \frac{\sum Rbj}{m},\quad \bar{R}c = \frac{\sum Rcj}{m} \qquad (7)$$

2-2- α -Cut fuzzy $\tilde{\bar{\bar{X}}}$ control charts based on ranges:

An α-cut is a non fuzzy set which comprises of all elements whose membership degrees are greater than or equal to α.

$$U\tilde{C}L^{\alpha}_{\bar{x}} = (\bar{\bar{X}}^{\alpha}_a, \bar{\bar{X}}_b, \bar{\bar{X}}^{\alpha}_c) + A_2(\bar{R}^{\alpha}_a, \bar{R}_b, \bar{R}^{\alpha}_c) \qquad (8)$$

$$\tilde{C}L = (\bar{\bar{X}}_a^{\alpha}, \bar{\bar{X}}_b^{\alpha}, \bar{\bar{X}}_c^{\alpha}) \qquad (9)$$

$$L\tilde{C}L^{\alpha}_{\bar{x}} = (\bar{\bar{X}}^{\alpha}_a, \bar{\bar{X}}_b, \bar{\bar{X}}^{\alpha}_c) - A_2(\bar{R}^{\alpha}_a, \bar{R}_b, \bar{R}^{\alpha}_c) \qquad (10)$$

where,

$$\bar{\bar{X}}^{\alpha}_a = \bar{\bar{X}}_a + \alpha(\bar{\bar{X}}_b - \bar{\bar{X}}_a), \qquad \bar{\bar{X}}^{\alpha}_c = \bar{\bar{X}}_c - \alpha(\bar{\bar{X}}_c - \bar{\bar{X}}_b)$$
$$\bar{R}^{\alpha}_a = \bar{R}_a + \alpha(\bar{R}_b - \bar{R}_a), \qquad \bar{R}^{\alpha}_c = \bar{R}_c - \alpha(\bar{R}_c - \bar{R}_b)$$

2-3- α -Level fuzzy midrange for α -Cut fuzzy $\tilde{\bar{\bar{X}}}$ control chart based on ranges:

α-Level fuzzy midrange is one of four transformation techniques used to determine the fuzzy control limits. These control limits are used to give a decision such as in-control or out-of-control for a process.

$$UCL^{\alpha}_{mr-\overline{X}} = CL^{\alpha}_{mr-\overline{X}} + A_2\left(\frac{\overline{R}^{\alpha}_{a} + \overline{R}^{\alpha}_{c}}{2}\right) \quad (11)$$

$$CL^{\alpha}_{mr-\overline{X}} = f^{\alpha}_{mr-\overline{X}}(C\widetilde{L}) = \frac{CL^{\alpha}_{(\overline{X})1} + CL^{\alpha}_{(\overline{X})3}}{2} \quad (12)$$

$$LCL^{\alpha}_{mr-\overline{X}} = CL^{\alpha}_{mr-\overline{X}} - A_2\left(\frac{\overline{R}^{\alpha}_{a} + \overline{R}^{\alpha}_{c}}{2}\right) \quad (13)$$

The condition of process control for each sample can be defined as:

$$\text{Process control} = \left\{ \begin{array}{ll} \text{in-control} & \text{for } LCL^{\alpha}_{mr-\overline{X}} \leqslant S^{\alpha}_{mr-\overline{X}_j} \leqslant UCL^{\alpha}_{mr-\overline{X}} \\ \text{out-of control} & \text{for otherwise} \end{array} \right\}. \quad (14)$$

where:

$$S^{\alpha}_{mr-\bar{x},j} = \frac{(\overline{X}_{a_j} + \overline{X}_{c_j}) + \alpha[(\overline{X}_{b_j} - \overline{X}_{a_j}) - (\overline{X}_{c_j} - \overline{X}_{b_j})]}{2} \quad (15)$$

2-4- Fuzzy $\widetilde{R}$ control chart:

Shewhart traditional R control chart is given by the following equation:

$$UCL_R = D_4\overline{R} \quad (16)$$

$$CL_R = \overline{R} \quad (17)$$

$$LCL_R = D_3\overline{R} \quad (18)$$

where D4 and D3 are control chart coefficients.

Fuzzy $\widetilde{R}$ control chart limits can be obtained in a similar way to traditional R control charts but they are represented by triangular fuzzy numbers as follows:

$$U\widetilde{C}L_R = D_4\overline{R} = D_4(\overline{R}_a, \overline{R}_b, \overline{R}_c) \quad (19)$$

$$C\widetilde{L}_R = \overline{R} = (\overline{R}_a, \overline{R}_b, \overline{R}_c) \quad (20)$$

$$L\widetilde{C}L_R = D_3\overline{R} = D_3(\overline{R}_a, \overline{R}_b, \overline{R}_c) \quad (21)$$

2-5- α-Cut fuzzy $\widetilde{R}$ control chart:

Control limits of α -cut fuzzy $\widetilde{R}$ control chart can be stated as follows :

$$U\widetilde{C}L^{\alpha}_{R} = D_4\overline{R}^{\alpha} \quad (22)$$

$$\widetilde{C}L^{\alpha}_{R} = \overline{R}^{\alpha} \quad (23)$$

$$L\widetilde{C}L^{\alpha}_{R} = D_3\overline{R}^{\alpha} \quad (24)$$

2-6- α-Level fuzzy midrange for α-Cut $\widetilde{R}$ control chart:

Control limits of α-level fuzzy midrange for α-cut fuzzy $\widetilde{R}$ control chart can be calculated as follows:

$$UCL^{\alpha}_{mr-R} = D_3 f^{\alpha}_{mr-R}(\widetilde{CL}) \quad (25)$$

$$CL^{\alpha}_{mr-R} = f^{\alpha}_{mr-R}(\widetilde{CL}) = \frac{\overline{R}^{\alpha}_{a} + \overline{R}^{\alpha}_{c}}{2} \quad (26)$$

$$LCL^{\alpha}_{mr-R} = D_4 f^{\alpha}_{mr-R}(\widetilde{CL}) \quad (27)$$

The condition of process control for each sample can be defined as:

$$\text{Process control} = \begin{cases} \text{in-control} & \text{for } LCL^{\alpha}_{mr-R} \leqslant S^{\alpha}_{mr-R,j} \leqslant UCL^{\alpha}_{mr-R} \\ \text{out-of control} & \text{for otherwise} \end{cases}. \quad (28)$$

where,

$$S^{\alpha}_{mr-R,j} = \frac{(R_{aj} + R_{cj}) + \alpha[(R_{bj} - R_{aj}) - (R_{cj} - R_{bj})]}{2} \quad (29)$$

3-Application:

The application was made on controlling the proportion of chlorine in the bottled water. Ten samples with a sample size of 4 (the total measurement number is 4×10=40) were taken from the production process in Ala Company.

3-1- Fuzzy $\overline{X}$ and R control chart:

Table (1): proportion of chlorine in bottled water for 10 days

samples	X_1	X_2	X_3	X_4	$\overline{X}$	R
1	21.3	23.07	19.52	15.97	19.96	7.1
2	17.75	24.07	23.07	17.75	20.85	7.1
3	19.52	19.52	19.52	17.75	19.07	1.77
4	23.07	17.75	21.3	23.07	21.29	5.32
5	23.07	19.52	15.97	24.85	20.85	8.88
6	19.52	23.07	17.75	19.52	19.96	5.32
7	14.2	17.75	21.3	23.07	19.08	8.87
8	21.3	17.75	19.52	17.75	19.08	3.55
9	23.07	21.3	17.75	21.3	20.85	5.32
10	23.07	15.95	19.52	17.75	19.07	7.12

$\overline{\overline{X}}$ = **20.006 ,** $\overline{R}$ **= 6.035 , for n=4 , A_2= 0.73**

Using equation (1, 2, 3):

UCL $\overline{X}$ =24.41 , CL $\overline{X}$ =20.006 , LCL $\overline{X}$ = 15.6

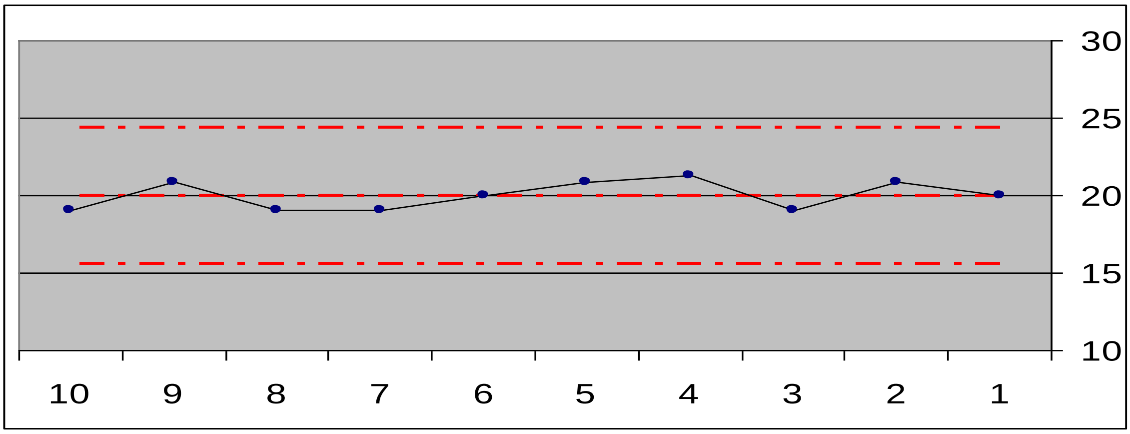

Fig.2. Shewhart $\overline{X}$ control chart based on rang

Appling $\overline{X}$ and R control chart, the process is in control.

Table (2):The fuzzy measurement values.

samples	X_a	X_b	X_c
S_{1-1}	17.75	21.3	24.85
S_{1-2}	21.30	23.07	24.85
S_{1-3}	17.75	19.52	21.3
S_{1-4}	14.2	15.97	17.75
S_{2-1}	10.65	17.75	24.85
S_{2-2}	21.3	24.85	28.4
S_{2-3}	21.3	23.07	24.85
S_{2-4}	14.2	17.75	21.3
S_{3-1}	17.75	19.95	21.3
S_{3-2}	17.75	19.95	21.3
S_{3-3}	17.75	19.95	21.3
S_{3-4}	14.2	17.75	21.3
S_{4-1}	21.3	23.07	24.85
S_{4-2}	14.2	17.75	21.3
S_{4-3}	17.75	21.3	24.85
S_{4-4}	21.3	23.07	24.85
S_{5-1}	21.3	23.07	24.85
S_{5-2}	17.75	19.52	21.3
S_{5-3}	14.2	15.97	17.75
S_{5-4}	21.3	24.85	28.4
S_{6-1}	17.75	19.52	21.3
S_{6-2}	21.3	23.07	24.85
S_{6-3}	14.2	17.75	21.3
S_{6-4}	14.2	19.52	24.85
S_{7-1}	10.65	14.2	17.75
S_{7-2}	14.2	17.75	21.3
S_{7-3}	17.75	21.3	24.85
S_{7-4}	21.3	23.07	24.85
S_{8-1}	17.75	21.3	24.85
S_{8-2}	14.2	17.75	21.3
S_{8-3}	14.2	19.52	24.85
S_{8-4}	14.2	17.75	21.3
S_{9-1}	21.3	23.07	24.85
S_{9-2}	17.75	21.3	24.85
S_{9-3}	14.2	17.75	21.3
S_{9-4}	17.75	21.3	24.85
S_{10-1}	21.3	23.07	24.85
S_{10-2}	14.2	15.97	17.75
S_{10-3}	17.75	19.52	21.3
S_{10-4}	14.2	17.75	21.3

Table (3): The fuzzy ranges and arithmetic means

samples	$\bar{X}_a$	$\bar{X}_b$	$\bar{X}_c$	R_a	R_b	R_c
1	17.75	19.96	22.18	3.55	7.1	10.65
2	16.86	20.85	24.85	0	7.1	17.75
3	16.86	19.07	21.3	-3.55	1.77	7.1
4	18.63	21.29	23.96	0	5.32	10.65
5	18.63	20.85	23.07	3.55	7.1	14.2
6	16.86	19.96	23.07	0	5.32	10.65
7	15.97	19.08	22.18	3.55	8.85	14.2
8	15.08	19.08	23.07	-3.55	3.55	10.65
9	17.75	20.85	23.96	0	5.32	10.65
10	16.86	19.07	21.3	0	7.1	10.65
average	17.125	20.006	22.894	.355	5.853	11.715

Using equation (4,5,6)

$$U\tilde{C}L_{\bar{X}} = (17.384, 24.279, 31.446)$$
$$\tilde{C}L_{\bar{X}} = (17.125, 20.006, 22.894)$$
$$L\tilde{C}L_{\bar{X}} = (8.573, 15.733, 22.635)$$

By using fuzzy $\bar{X}$ control chart based on ranges, we obtain the result that the process is in control.

3-2- α-Cut fuzzy $\tilde{\bar{X}}$ control chart based on ranges:

$$\bar{\bar{X}}_a^{\alpha} = \bar{\bar{X}}_a + \alpha(\bar{\bar{X}}_b - \bar{\bar{X}}_a) = 18.5655 \qquad \bar{\bar{X}}_c^{\alpha} = \bar{\bar{X}}_c - \alpha(\bar{\bar{X}}_c - \bar{\bar{X}}_b) = 21.45$$
$$\bar{R}_a^{\alpha} = \bar{R}_a + \alpha(\bar{R}_b - \bar{R}_a) = 3.104 \qquad \bar{R}_c^{\alpha} = \bar{R}_c - \alpha(\bar{R}_c - \bar{R}_b) = 8.784$$

Using equation (8, 9, 10) :

$$U\tilde{C}L^{\alpha}_{\bar{X}} = (20.83, 24.27, 27.86)$$
$$\tilde{C}L = (18.565, 20.006, 27.86)$$
$$L\tilde{C}L^{\alpha}_{\bar{X}} = (12.152, 15.733, 19.184)$$

By using α -Cut we get the same result. The process is in control.

3-3-α-Level fuzzy midrange for α-Cut fuzzy $\tilde{\bar{X}}$ control chart based on ranges:

Using equation (11, 12, 13):

$UCL^{\alpha}_{mr-\bar{X}}$ =24.34 , $CL^{\alpha}_{mr-\bar{X}}$ =20.007 , $LCL^{\alpha}_{mr-\bar{X}}$ =15.66

Using equation (14, 15) we get our decision as follows in table(4):

Table (4): The decision using α-Cut fuzzy $\tilde{\tilde{X}}$ control chart based on ranges

samples	$S^{\alpha}_{mr-\bar{X}\,i}$	Decision
1	19.96	In control
2	20.85	In control
3	19.07	In control
4	21.29	In control
5	20.85	In control
6	19.96	In control
7	19.08	In control
8	19.08	In control
9	20.85	In control
10	10.07	In control

3-4- Fuzzy $\tilde{R}$ control chart:

Shewhart traditional R control chart using equation (16, 17, 18)
For n=4, D_4= 2.28 and D_3=0

UCL_R=13.76 , CL_R= 6.035 , LCL_R= 0

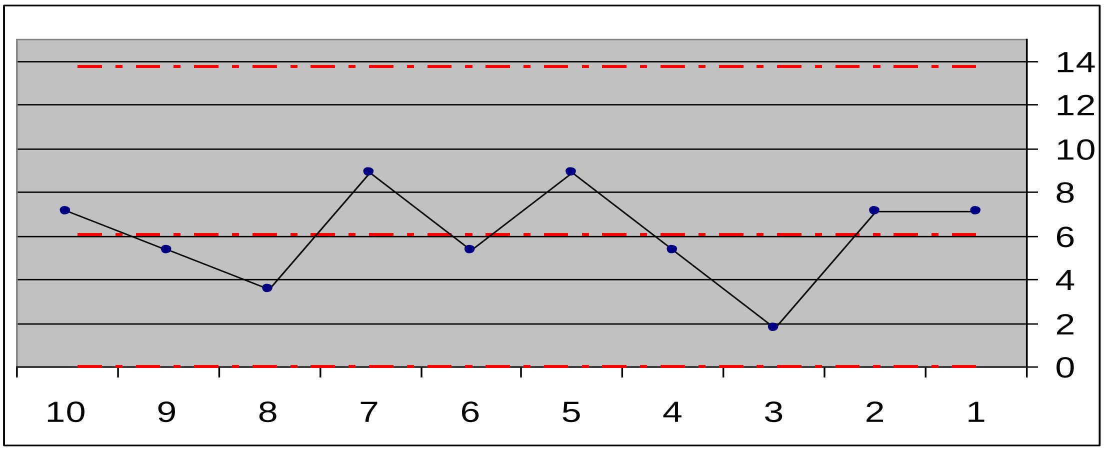

Fig.3. Shewhart R control chart

Using R control chart, the process is in control.

Fuzzy R control chart using equation (19, 20, 21):

$$U\tilde{C}L_R = (0.809, 13.34, 26.71)$$
$$\tilde{C}L_R = (0.33, 5.853, 11.715)$$
$$L\tilde{C}L_R = (0, 0, 0)$$

The process is in control.

3-5- α-Cut fuzzy $\tilde{R}$ control chart:

Using equation (22, 23, 24):

$$U\tilde{C}L^{\alpha}_R = (7.07, 13.34, 20.02)$$
$$\tilde{C}L^{\alpha}_R = (3.104, 5,85, 8.784)$$
$$L\tilde{C}L^{\alpha}_R = (0, 0, 0)$$

The process is in control.

3-6- α-Level fuzzy midrange for α-Cut $\tilde{R}$ control chart:

Using equation(25, 26, 27):

$UCL^{\alpha}_{mr-R} = 13.55$, $CL^{\alpha}_{mr-R} = 5.944$, $LCL^{\alpha}_{mr-R} = 0$

Using equation (28, 29) we get our decision as follows in table (5):

Table (5): The decision using α-Cut $\tilde{R}$ control chart

samples	S^{α}_{mr-Rj}	Decision
1	7.1	In control
2	7.98	In control
3	1.77	In control
4	5.32	In control
5	7.98	In control
6	5.32	In control
7	8.86	In control
8	3.55	In control
9	5.32	In control
10	6.21	In control

4-Conclusion:

With this paper, it is shown that fuzzy set theory is applicable on traditional variable control charts. Fuzzy observations and fuzzy control charts can provide more flexibility for controlling process and have more appropriate mathematical description frame than control chart approach and give more meaning results than traditional quality control charts.

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